

Calculation of Graviton Scattering Amplitudes using String-Based Methods

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Abstract

Techniques based upon the string organisation of amplitudes may be used to simplify field theory calculations. We apply these techniques to perturbative gravity and calculate all one-loop amplitudes for four-graviton scattering with arbitrary internal particle content. Decomposing the amplitudes into contributions arising from supersymmetric multiplets greatly simplifies these calculations. We also discuss how unitarity may be used to constrain the amplitudes.

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1. Introduction

Calculations in perturbative gravity are well known to be prohibitively difficult using conventional Feynman diagram techniques [1]. In typical gauges, the graviton vertices contain considerably more terms than in gauge theories and two powers of loop momentum rather than one. These two features help to make perturbative gravity calculation an algebraic nightmare.

Recently, the alternate organisation of amplitudes offered by string theory has been used by Bern and Kosower to construct rules for computations in gauge theories [2,3]. These rules allow a considerable algebraic simplification compared to normal Feynman diagram techniques as evidenced by the first calculation of the five-gluon one loop amplitudes [4]. Since string theory also contains gravity one can apply these techniques to obtain rules for calculations in perturbative gravity. In ref. [5], the string-based technique for perturbative gravity was outlined and a sample calculation of the $A^{\text{one-loop}}(-, +, +, +)$ four graviton helicity amplitude was performed. This is the simplest of the four graviton amplitudes being finite and without cuts and has been previously calculated by first calculating the contributions to the four graviton amplitude from real scalars in the loop and then using the supersymmetry Ward identities [6].

In this paper we present a detailed description of the rules for one-loop n -graviton amplitudes and use these to calculate the four graviton amplitude for all helicity configurations with arbitrary particle content in the loop. As with the QCD method, these rules arise by looking at the infinite tension limit [7,8] of a string theory. However, these rules can be used with no knowledge of string theory. The rules are in many ways a “double” copy of those for QCD,

$$(\text{Gravity}) \sim (\text{Yang-Mills})^2 . \quad (1.1)$$

which reflects the fact that in string theory, a closed string may be regarded as the product of two open strings [9]

$$(\text{Closed String}) \sim (\text{Open String})^2 . \quad (1.2)$$

This equivalence is largely true at the level of the integrands of diagrams. (There is, however, a small amount of interference between the two “halves” of the string which is related to the zero mode integral in string theory.) Using tree-level relationships which embody (1.1) [10] string theory has been used to calculate tree-level graviton amplitudes previously in ref. [11].

As inspired by the string-based method, we implement a supersymmetric decomposition of the amplitudes similar to that recently used for gauge theories [12,13]. Instead of calculating the contributions to the loop amplitudes from individual particles circulating (which may be the graviton, gravitino, vector, Weyl fermion or scalar in a gravitational theory) we calculate the contributions from various supersymmetric multiplets plus the scalar

contribution. Corresponding to the five particle types we must calculate four supersymmetric contributions (which we choose to be the $N = 1$, $N = 4$, $N = 6$ and $N = 8$ matter multiplets) plus the scalar. The contribution from any individual particle type is just a linear combination of these. This decomposition enables us to exploit the simplifications found in supersymmetry calculations. In supersymmetric theories there are cancellations between the fermions and bosons. If one uses a suitable formalism, these cancellations are manifest diagram by diagram. This proves to be an enormous simplification. Examples of such beautiful formalisms are 1) the string based rules and 2) a superspace formalism using the background field method [14,15,16]. In a general gauge even in a superspace formalism the cancellations do not occur diagram by diagram. The relationship between string based rules for gauge theories and conventional field theory is particularly close when the field theory is organised using the background field method [17,18].

For the four-point function, without cancellations, the Feynman parameter integral generically has eight powers of Feynman parameters (or equivalently eight powers of loop momentum). For the $N = 1$ matter multiplet (containing a scalar and a fermion) there is a cancellation of the two leading powers of Feynman parameters which simplifies the calculation considerably. With increasing N more cancellations occur until for the maximal case, $N = 8$, all eight powers cancel and one is left with a trivial sum of scalar box integrals. The special cases of $N = 8$ supergravity and $N = 4$ super-Yang-Mills four point functions were obtained previously in ref. [19] also by examining the infinite string tension limit.

Another advantage of the supersymmetric decomposition is that the supersymmetric amplitudes, again due to the cancellations in loop momentum, can be strongly constrained by unitarity via the Cutkosky rules [20]. In ref. [21,22], situations in a gauge theory where unitarity completely determines the one-loop amplitudes are given. For the four-point gravity calculation the $N = 8$ and $N = 6$ contributions can be determined completely using the Cutkosky rules and are in agreement with the string-based results. The Cutkosky rules are also used to check the cuts in the remaining amplitudes. As a further example of the uses of the Cutkosky rules we calculate the logarithmic part of the two-loop amplitude $A^{2\text{-loop}}(++++)$.

As is well known, pure gravity is renormalisable at one-loop [23] whereas gravity coupled to matter is not [24]. However, the ultra-violet infinities do not arise in amplitudes containing only external gravitons but appear in amplitudes with external matter [25]. Our amplitudes are all ultra-violet finite in agreement with the formal arguments. The amplitude with gravitons circulating in the loop contain the infra-red singularities as expected.

2. Rules for one-loop gravity

In ref. [2,3] rules were introduced for the calculation of gauge theory amplitudes. These

were obtained by taking the infinite string tension limit of string theory amplitudes and can be used instead of Feynman diagram techniques. Although derived from string theory, they can be used without explicit knowledge of string theory. The various contributions to the amplitude are associated with ϕ^3 diagrams. Typically, the string organisation lead to a more compact integrand for these diagrams than that arising in conventional field theory. The string-based technique has been used to perform significant calculations such as the five-gluon one-loop contributions [4].

There are two slightly different but equivalent formulations of the rules; One is obtained by taking the infinite tension limit of a superstring [2,3] whereas the other is obtained by taking the infinite tension limit of a bosonic string [26,12]. The bosonic form of the rules is more compact but the appropriate rules for fermions circulating in the loop must be inferred from the superstring case. The string-based rules for one-loop gravity which we present here were outlined in ref. [5]. These are based upon the bosonic formulation. The rules for gravity have many similarities to those for gauge theories so we will be brief in presenting them.

The initial step in the rules is to draw all ϕ^3 diagrams, excluding tadpoles. There is also no need to include diagrams with a loop isolated on an external leg since these vanish when dimensional regularisation is used. The external legs of these diagrams should be labeled, with diagrams containing all orderings included. The inner lines of trees attached to the loop are labeled according to the rule that as one moves from the outer lines to the inner ones, one labels the inner line with the same label as the most clockwise of the two outer lines attached to it. The contribution from each labeled n -point ϕ^3 -like diagram with n_ℓ legs attached to the loop is

$$\mathcal{D} = i \frac{(-\kappa)^n}{(4\pi)^{2-\epsilon}} \Gamma(n_\ell - 2 + \epsilon) \int_0^1 dx_{i_{n_\ell-1}} \int_0^{x_{i_{n_\ell-1}}} dx_{i_{n_\ell-2}} \cdots \int_0^{x_{i_3}} dx_{i_2} \int_0^{x_{i_2}} dx_{i_1} \times \frac{K_{\text{red}}(x_{i_1}, \dots, x_{i_{n_\ell}})}{\left(\sum_{l < m}^{n_\ell} P_{i_l} \cdot P_{i_m} x_{i_m i_l} (1 - x_{i_m i_l}) \right)^{n_\ell - 2 + \epsilon}} \quad (2.1)$$

where the ordering of the loop parameter integrals corresponds to the ordering of the n_ℓ lines attached to the loop, $x_{ij} \equiv x_i - x_j$. The x_{i_m} are related to ordinary Feynman parameters by $x_{i_m} = \sum_{j=1}^m a_j$. K_{red} is the “reduced kinematic factor”, which the string-based rules efficiently yield in a compact form. The lines attached to the loop carry momenta P_i which will be off-shell if there is a tree attached to that line. The dimensional regularisation parameter $2\epsilon = 4 - D$ handles all ultra-violet and infra-red divergences. The amplitude is then given by summing over all diagrams. We also use the equivalent

Schwinger proper-time form of the amplitude

$$\mathcal{D}_S = i \frac{(-\kappa)^n}{(4\pi)^{2-\epsilon}} \int \prod_{i=1}^{n_\ell-1} dx_i \prod_{i<j}^{n_\ell} \int_0^\infty dT T^{n_\ell-3+\epsilon} \exp\left(-T \sum_{l<m}^{n_\ell} P_{i_l} \cdot P_{i_m} x_{i_m i_l} (1 - x_{i_m i_l})\right) \times K_{\text{red}}(x_{i_1}, \dots, x_{i_{n_\ell}}, T) \quad (2.2)$$

as discussed in ref.[17]

In order to evaluate K_{red} , one starts with the graviton kinematic expression

$$\mathcal{K} = \int \prod_{i=1}^n dx_i d\bar{x}_i \prod_{i<j}^n \exp\left[k_i \cdot k_j G_B^{ij}\right] \exp\left[(k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) \dot{G}_B^{ij} - \varepsilon_i \cdot \varepsilon_j \ddot{G}_B^{ij}\right] \times \exp\left[(k_i \cdot \bar{\varepsilon}_j - k_j \cdot \bar{\varepsilon}_i) \dot{\bar{G}}_B^{ij} - \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \ddot{\bar{G}}_B^{ij}\right] \exp\left[-(\varepsilon_i \cdot \bar{\varepsilon}_j + \varepsilon_j \cdot \bar{\varepsilon}_i) H_B^{ij}\right] \Big|_{\text{multi-linear}} \quad (2.3)$$

where the ‘multi-linear’ indicates that only the terms linear in all ε_i and $\bar{\varepsilon}_i$ are included. The graviton polarization tensor is reconstructed by taking $\varepsilon_i^\mu \bar{\varepsilon}_i^\nu \rightarrow \varepsilon_i^{\mu\nu}$. From a string theory perspective G_B is the bosonic Green function on the string world sheet, \dot{G}_B and \ddot{G}_B are derivatives of this Green function with respect to left-moving variables, and $\dot{\bar{G}}_B$ and $\ddot{\bar{G}}_B$ are derivatives with respect to right-movers. (Since a closed string is periodic the variables described the string world sheet can split into “left-moving” and “right-moving”.) The term H_B^{ij} is the derivative of the Green function with respect to one left mover and one right mover variable. The functions G_B^{ij}, \dot{G}_B^{ij} and H_B^{ij} are to taken as symmetric in the i and j indices while \dot{G}_B is antisymmetric. Although the above expression contains much information in string theory, when one takes the infinite string tension limit [2,3] it should merely be regarded as a function which contains all the information necessary to generate K_{red} for all graphs. The utility of the string based method partially lies in this compact representation (which is valid for arbitrary numbers of legs!). The existence of an overall function which reduces to the Feynman parameter polynomial for each diagram is one of the most useful features of the string based rules.

The appropriate expression for gauge theories is obtained by setting $\bar{\varepsilon} = 0$ in the above. The gravity expression is like a double copy of the gauge theory expression apart from the H_B^{ij} terms which mix the left and right movers.

The first step in applying the rules is to remove all of the $\ddot{\bar{G}}_B^{ij}$ and \ddot{G}_B^{ij} by integrating the kinematic expression by parts with respect to the variables x_i and \bar{x}_i where necessary. When manipulating this formula we take $\dot{G}_B^{ij}, \ddot{G}_B^{ij}, \dot{\bar{G}}_B^{ij}$ and $\ddot{\bar{G}}_B^{ij}$ to mean $\partial_{x_i} G_B^{ij}, \partial_{x_i}^2 G_B^{ij}, \partial_{\bar{x}_i} G_B^{ij}$ and $\partial_{\bar{x}_i}^2 G_B^{ij}$ respectively. (After direct substitution of the values of the functions in the field theory limit these relations are almost but not quite true; however, for the purposes of manipulating eq. (2.3) this distinction is unimportant.) While carrying out

this process one must take into account the cross-terms where a left-mover derivative hits right-mover terms, and vice versa. This can be done by using the results

$$\begin{aligned}\frac{\partial}{\partial \bar{x}_k} \dot{G}_B^{ij} &= \delta_{ki} H_B^{ij} - \delta_{kj} H_B^{ij} & \frac{\partial}{\partial x_k} \dot{\bar{G}}_B^{ij} &= \delta_{ki} H_B^{ij} - \delta_{kj} H_B^{ij} \\ \frac{\partial}{\partial \bar{x}_k} \ddot{G}_B^{ij} &= 0 & \frac{\partial}{\partial x_k} \ddot{\bar{G}}_B^{ij} &= 0\end{aligned}$$

For example, if the expression

$$\int \prod_{i=1}^4 dx_i d\bar{x}_i \prod_{i<j}^4 \exp \left[k_i \cdot k_j G_B^{ij} \right] \dot{G}_B^{34} \ddot{G}_B^{12} \dot{\bar{G}}_B^{13} (\dot{\bar{G}}_B^{34})^2$$

is integrated by parts with respect to x_1 , the result is

$$\begin{aligned}\int \prod_{i=1}^4 dx_i d\bar{x}_i \prod_{i<j}^4 \exp \left[k_i \cdot k_j G_B^{ij} \right] \dot{G}_B^{34} \dot{G}_B^{12} (\dot{\bar{G}}_B^{34})^2 \\ \times \left((k_1 \cdot k_2 \dot{G}_B^{12} + k_1 \cdot k_3 \dot{G}_B^{13} + k_1 \cdot k_4 \dot{G}_B^{14}) \dot{\bar{G}}_B^{13} + H_B^{13} \right).\end{aligned}$$

Having carried out the integration by parts we now may carry out simple substitution rules for each diagram to obtain K_{red} . First the $\prod_{i<j}^n \exp[k_i \cdot k_j G_B^{ij}]$ term and the integrals over x_i and \bar{x}_i are dropped from the kinematic expression. (Since the appropriate contributions have been included in the rules). After integration by parts, \mathcal{K} will be a sum of terms each of which has n \dot{G}_B and n $\dot{\bar{G}}_B$. (An H_B is equivalent to one \dot{G}_B and one $\dot{\bar{G}}_B$.)

Any diagram will be a loop with n_ℓ legs attached with possible non-trivial trees attached to the loop. The rules have two parts. Firstly tree rules are applied to \mathcal{K} . These produce a truncated \mathcal{K} which is a series of terms each with n_ℓ \dot{G}_B and $\dot{\bar{G}}_B$. Secondly loop substitution rules are applied which give the Feynman parameter polynomial for the diagram. The tree rules are applied iteratively working from the outside of the attached trees towards the loop. For a two-point tree with outer legs labeled by i and j , one carries out the substitutions

$$\begin{aligned}(\dot{G}_B^{ij})^n (\dot{\bar{G}}_B^{ij})^m &\rightarrow \delta_{n,1} \delta_{m,1} \frac{1}{(-2k_i \cdot k_j)} \\ i \rightarrow j &\quad \text{in remaining factors}\end{aligned} \tag{2.4}$$

in each term. This should be applied at each tree vertex.

Once the tree rules have been carried out for a diagram, one applies the loop rules. These depend on the particles circulating in the loop. They are essentially independent

applications of the Yang-Mills rules to the left- and right-mover parts, with an extra substitution for cross-terms H_B .

One-loop amplitudes depend upon the particle circulating in the loop. and the substitution rules are corresponding different for different particle types. For gauge theories there are three types of particles/rules, those for scalars S , fermions F and vectors V . For gravity, the rules are a double copy of the gauge theory rules and this choice of substitutions can be chosen differently for the two copies. That is, we can apply different substitution rules to the \dot{G}_B (left-movers) and the $\vec{\dot{G}}_B$ (right movers). The particle content circulating in the loop corresponding to these choices of the loop substitution rules is given in table 1.

Substitution	Particle Content
$2[S, S]$	complex scalar
$-2[S, F]$	Weyl Fermion
$2[S, V]$	Vector
$-4[V, F]$	gravitino and Weyl Fermion
$4[V, V]$	graviton and complex scalar
$4[V, V] - 2[S, S]$	graviton
$-4[V, F] + 2[S, F]$	gravitino

Table 1: Applying the substitution rules shown corresponds to having the particle content shown circulating in the loop. $[x, y]$ denotes applying substitution rules x and y to \dot{G}_B and $\vec{\dot{G}}_B$.

F and V each produce two types of contribution. The first contribution is just the scalar S but the second is different in the two cases. The different contribution we refer to as the “cycle” contribution C_V and C_F .

$$\begin{aligned} F &= S + C_F \\ V &= S + C_V \end{aligned} \tag{2.5}$$

The common S contribution is obtained by making the substitutions

$$\begin{aligned} \dot{G}_B^{ij} &\longrightarrow \frac{1}{2}(-\text{sign}(x_{ij}) + 2x_{ij}) \\ \vec{\dot{G}}_B^{ij} &\longrightarrow \frac{1}{2}(-\text{sign}(x_{ij}) + 2x_{ij}) \\ H_B^{ij} &\longrightarrow \frac{1}{2T} \end{aligned} \tag{2.6}$$

in the Schwinger parameterization (2.2). (Before taking the infinite tension limit a δ -function exist in H_B^{ij} however as discussed in ref. [8] this δ -function does not contribute

in the infinite string tension limit of physical amplitudes.) The cycle contribution comes from “cycles” of \dot{G}_B . A cycle is a sequence of \dot{G}_B ’s

$$\dot{G}_B^{i_1 i_2} \dot{G}_B^{i_2 i_3} \dots \dot{G}_B^{i_m i_1}$$

The substitution rules for these cycles is different in the three cases. For the scalar they are vanishing. For C_V , the substitution rules are

$$\begin{aligned} \dot{G}_B^{i_1 i_2} \dot{G}_B^{i_2 i_1} &\rightarrow 1 \\ \dot{G}_B^{i_1 i_2} \dot{G}_B^{i_2 i_3} \dots \dot{G}_B^{i_{m-1} i_m} \dot{G}_B^{i_m i_1} &\rightarrow 1/2 \quad (m > 2) \end{aligned}$$

where all the cycles must follow the ordering of the legs, and only one cycle at a time may contribute to any term. Once these substitutions have been made all remaining \dot{G}_B ’s should be replaced as in eq. (2.6). For C_F the following substitution is made

$$\dot{G}_B^{i_1 i_2} \dot{G}_B^{i_2 i_3} \dots \dot{G}_B^{i_{m-1} i_m} \dot{G}_B^{i_m i_1} \rightarrow -(-1/2)^m \prod_{k=1}^m \text{sign}(x_{i_k i_{k+1}})$$

In contrast to the V rules, all cycles contribute in the F case regardless of ordering. Also, all combinations of one or more cycles from each term contribute. Again, once these substitutions have been made all remaining \dot{G}_B ’s should be replaced as in (2.6).

For example if, for the four-point amplitude, we have a term in the Kinematic expression

$$K = (\dot{G}_B^{12})^2 (\dot{G}_B^{34})^2 \quad (2.7)$$

Then for the box diagram with ordering of legs 1234 the cycle contributions for the two cases are

$$\begin{aligned} C_F : K &\longrightarrow -\frac{1}{4} \left(\frac{1}{2} (1 + 2x_{34}) \right)^2 - \frac{1}{4} \left(\frac{1}{2} (1 + 2x_{12}) \right)^2 + \frac{1}{16} \\ C_V : K &\longrightarrow \left(\frac{1}{2} (1 + 2x_{34}) \right)^2 + \left(\frac{1}{2} (1 + 2x_{12}) \right)^2 \end{aligned} \quad (2.8)$$

there being no cycle contribution in the scalar case.

This process gives an expression for K_{red} for each diagram for arbitrary particle content in the loop. The integral in (2.1) can now be carried out. The contributions from each diagram are then summed over. Explicit simple examples of the applications of the string based rules for QCD are given in refs. [3,12] and for gravity in ref. [5] which the interested reader may wish to examine to see the simplicity of the string based method. The string based rules have advantages in producing compact expressions for the numerators in the Feynman parameter integrals.

3. Supersymmetric Decomposition

A useful way to organise the n -graviton amplitudes is to use a “supersymmetric decomposition”. The loop amplitudes depends upon the state circulating in the loop. In a graviton scattering calculation this may be one of five states: a scalar, a Weyl fermion, a vector, a gravitino or the graviton itself. The string based rules can be used to calculate these contributions individually. However, it proves convenient to calculate the contributions from supersymmetric multiplets instead. Amplitudes for all choices of particles in the loop can be written as linear combinations of those for certain choices of supersymmetric multiplets and for a scalar. In particular we choose one multiplet from each of $N=1,4,6,8$ supersymmetric theories with particle content given in the following table. These multiplets are centered around the spin-0 complex scalar.

N	scalars	spin-1/2	spin-1	spin-3/2	spin-2
$N = 0$	1				
$N = 1$	1	1			
$N = 4$	3	4	1		
$N = 6$	10	15	6	1	
$N = 8$	35	56	28	8	1

Table 2: Particle content of the supersymmetric multiplets we consider. Scalars are complex, and the fermions are Weyl.

This decomposition is useful when evaluating amplitudes. In general, the integrations involved in amplitudes increases considerable with the degree of the polynomial in the numerator of the Feynman parameter integral. (Or equivalently with the degree of the loop momentum polynomial if performing momentum integrals.) If one uses the string-based rules cancellations due to supersymmetry occur within each diagram, reducing the complexity of computations. Similar cancellations occur if one uses a background field method within a superfield formalism [16]. (The relationship between background field methods and string based calculations is explored in [17].)

Specifically, within the string based rules these simplifications can be seen as cancellations between the common contributions within multiplets. In terms of Feynman parameters, for a general n -point integral the scalar term S is a polynomial of degree n however the cycle contributions are polynomials of degree $n - 2$. For each particle type, there is a scalar contribution $N_s[S, S]$ where N_s counts the degrees of freedom with fermions having negative weight. Hence for any supersymmetric multiplet the $[S, S]$ term will cancel and the Feynman parameter polynomial will be simplified. With increasing N there are increasing cancellations. Also for the combination $C_V - 4C_F$ the two- and three-cycle contributions cancel leaving a polynomial of degree $n - 4$. This may be seen, for example,

by comparing the cycle contributions in eqn. (2.8). For the supergravity multiplets cancellations can occur on both left and right movers. For example, for the $N = 8$ calculation we have

$$A^{N=8} = A^{\text{graviton}} - 8A^{\text{gravitino}} + 28A^{\text{vector}} - 56A^{\text{fermion}} + 30A^{\text{scalar}} \quad (3.1)$$

Inserting the rules from table 1 we find

$$A^{N=8} = 4[C_V, C_V] - 32[C_V, C_F] + 64[C_F, C_F] = 4[C_V - 4C_F, C_V - 4C_F] \quad (3.2)$$

From this we see that for a n -point integral the Feynman parameter polynomial would be $2n - 8$ at most. The cancellations for a given N are shown in table 3. (We are using a regularisation scheme which preserves supersymmetry [27] which simplifies the decomposition. In other regularisation schemes the form of the decomposition is a little more complex.)

N	Contribution	Degree
$N = 0$	$2[S, S]$	$2n$
$N = 1$	$2[C_F, S]$	$2n - 2$
$N = 4$	$2[C_V - 4C_F, S]$	$2n - 4$
$N = 6$	$-4[C_V - 4C_F, C_F]$	$2n - 6$
$N = 8$	$4[C_V - 4C_F, C_V - 4C_F]$	$2n - 8$

Table 3: The String rules appropriate for the multiplet are given and the degree of the Feynman parameter polynomial for an n -point loop integral.

To reconstruct the amplitudes for specific particles in the loop we can use

$$\begin{aligned}
A^{[0]} &= A^{N=0} \\
A^{[1/2]} &= A^{N=1} - A^{[0]} \\
A^{[1]} &= A^{N=4} - 4A^{N=1} + A^{[0]} \\
A^{[3/2]} &= A^{N=6} - 6A^{N=4} + 9A^{N=1} - A^{[0]} \\
A^{[2]} &= A^{N=8} - 8A^{N=6} + 20A^{N=4} - 16A^{N=1} + A^{[0]}
\end{aligned} \quad (3.3)$$

4. Four-graviton amplitudes

We now present the one-loop 4-graviton results for all choices of helicity. We will give each result in the supersymmetric decomposition form as described in the previous section. We also quote the pure gravity results explicitly.

The first step is to insert spinor helicity simplifications into the kinematic expression (2.3). The spinor helicity method for gravitons [11,28] is related to that for vectors [29] by

$$\varepsilon^{++} = \varepsilon^+ \bar{\varepsilon}^+, \quad \varepsilon^{--} = \varepsilon^- \bar{\varepsilon}^-$$

where $\varepsilon^{\pm\pm}$ are the graviton helicity polarizations and ε^\pm are the vector helicity polarizations defined by Xu, Zhang and Chang. We use the notation for spinor inner products $\langle k_1^- | k_2^+ \rangle = \langle 12 \rangle$ and $\langle k_1^+ | k_2^- \rangle = [12]$. The use of spinor helicity techniques has proved extremely useful in QCD calculation. All states are taken to be outgoing and may have plus or minus helicity. There is no concept of colour ordering which is found in QCD amplitudes. There are thus three independent helicity configurations for the four point amplitude, $(+, +, +, +)$, $(-, +, +, +)$ and $(-, -, +, +)$, the others being obtained by conjugation from these.

For the $(-, +, +, +)$ and $(+, +, +, +)$ one-loop amplitudes all the supersymmetric components in the decomposition vanish due to supersymmetric Ward identities analogous to the situation in QCD [30]. The tree level graviton amplitudes vanish for these helicity configurations and hence the one-loop results are the leading order for these configurations and have a simple form rather analogous to a tree amplitude without logarithms or infinities. From the inverse decomposition (3.3) this amplitude for any particle content is just proportional to the scalar contribution. The scalar contributions to graviton scattering have been previously calculated in ref. [6] but not in a spinor helicity basis. The results from the string-based rules agree with these results and we have checked explicitly that the cycle contributions cancel, demonstrating the Ward identities. We find

$$A(1^-, 2^+, 3^+, 4^+) = N_s \frac{i\kappa^4}{(4\pi)^2} \left(\frac{st}{u} \right)^2 \left(\frac{[24]^2}{[12] \langle 23 \rangle \langle 34 \rangle [41]} \right)^2 \frac{(s^2 + st + t^2)}{5760} \quad (4.1)$$

$$A(1^+, 2^+, 3^+, 4^+) = -N_s \frac{i\kappa^4}{(4\pi)^2} \left(\frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 \frac{(s^2 + st + t^2)}{1920} \quad (4.2)$$

where

$$N_s = N_B - N_F$$

is the number of bosonic states in the loop minus the number of fermionic states and $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$ and $u = (k_1 + k_3)^2$. So, for instance, since a graviton is made up of two helicity states the amplitudes for pure gravity are found by putting $N_s = 2$ in the above expressions.

For the $A(1^-, 2^-, 3^+, 4^+)$ amplitude, none of the cycle terms vanish, so these must be included. We express their contributions using the supersymmetric decomposition given

in the previous section. The (complex) scalar amplitude is

$$A^{[0]}(1^-, 2^-, 3^+, 4^+) = + \frac{F(t-u)(t^4 + 9ut^3 + 46u^2t^2 + 9u^3t + u^4) \ln(-t/-u)}{30s^7} \\ + \frac{F(2t^4 + 23ut^3 + 222u^2t^2 + 23u^3t + 2u^4)}{180s^6} - \frac{Fu^3t^3(\ln(-t/-u)^2 + \pi^2)}{s^8} \quad (4.3)$$

where F is

$$\frac{i\kappa^4(4\pi)^\epsilon r_\Gamma}{16(4\pi)^2} \left(\frac{st \langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 = \frac{ist u \kappa^2 (4\pi)^\epsilon r_\Gamma}{4(4\pi)^2} A^{\text{tree}}(1^-, 2^-, 3^+, 4^+). \quad (4.4)$$

$A^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$ is defined in eq. (5.3) and

$$r_\Gamma = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}. \quad (4.5)$$

The amplitudes for the supersymmetric multiplets given in table 2 are

$$A^{N=1} = -\frac{F(t^2 + 14tu + u^2)}{24s^4} + \frac{Ft^2u^2(\ln^2(-t/-u) + \pi^2)}{2s^6} \\ - \frac{F(t-u)(t^2 + 8tu + u^2) \ln(-t/-u)}{12s^5} \\ A^{N=4} = \frac{F}{2s^4} \left((t-u)s \ln(-t/-u) - tu(\ln^2(-t/-u) + \pi^2) + s^2 \right) \\ A^{N=6} = \frac{-F}{2} \left(\frac{\ln^2(-t/-u) + \pi^2}{s^2} \right) \\ A^{N=8} = \frac{2F}{\epsilon} \left(\frac{\ln(-u)}{st} + \frac{\ln(-t)}{su} + \frac{\ln(-s)}{tu} \right) \\ + 2F \left(\frac{\ln(-t) \ln(-s)}{st} + \frac{\ln(-u) \ln(-t)}{tu} + \frac{\ln(-s) \ln(-u)}{us} \right) \quad (4.6)$$

We chose to express the amplitude in the (unphysical) regime where all momentum variables s , t and u are negative. One can obtain expressions in the physical region by the substitution

$$\ln(-s) \rightarrow \ln(|s|) - i\pi\Theta(s) \quad (4.7)$$

etc. $\Theta(s)$ is the Heavyside function where $\Theta(x) = 1, x > 0$ and $\Theta(x) = 0, x < 0$.

The pure gravity amplitude can be found using the expression in eq. (3.3) which gives

the result

$$\begin{aligned}
A^{[2]}(1^-, 2^-, 3^+, 4^+) = F & \left(\frac{2}{\epsilon} \left(\frac{\ln(-u)}{st} + \frac{\ln(-t)}{su} + \frac{\ln(-s)}{tu} \right) \right. \\
& + \frac{2 \ln(-u) \ln(-s)}{su} + \frac{2 \ln(-t) \ln(-u)}{tu} + \frac{2 \ln(-t) \ln(-s)}{ts} \\
& + \frac{(t+2u)(2t+u)(2t^4+2t^3u-t^2u^2+2tu^3+2u^4)(\ln^2(-t/-u)+\pi^2)}{s^8} \\
& + \frac{(t-u)(341t^4+1609t^3u+2566t^2u^2+1609tu^3+341u^4)\ln(-t/-u)}{30s^7} \\
& \left. + \frac{1922t^4+9143t^3u+14622t^2u^2+9143tu^3+1922u^4}{180s^6} \right)
\end{aligned} \tag{4.8}$$

These expressions have the correct symmetry expected in the amplitude under, for example, interchange of legs 3 and 4.

Only the $N = 8$ amplitude has $1/\epsilon$ singularities. These are purely IR singularities the result being UV finite. This is as expected since the $N = 8$ multiplet is the only multiplet containing gravitons and the loop amplitudes with other particles circulating are expected to be IR finite.

5. Unitarity constraints & consistency checks

Unitarity, in the form of the Cutkosky rules, is a strong constraint on amplitudes. When cancellations occur in the loop momentum integrals it becomes particularly restricting and if enough cancellations occur unitarity may be enough to uniquely determine the amplitude. Again, the supersymmetric decomposition is useful in this context since cancellations occur within supersymmetry multiplets. In ref [22] it was proved that, for a gauge theory amplitude, if the n -point loop integral has at most $n - 2$ powers of loop momentum in the numerator the amplitude is uniquely determined from the cuts. Unfortunately, since gravity amplitudes generally have $2n$ powers of loop momentum this result is not as useful in determining amplitudes since for large n the power of the loop momentum polynomial grows to be larger than $n - 2$ even in the case of maximal cancellation $N = 8$. In this case we may apply this result, in principle, for $2n - 8 \leq n - 2$ that is for $n \leq 6$ to completely determine the amplitude. Looking at table 3 we see that for the four point amplitude this result may be applied to the $N = 8$ and $N = 6$ multiplet contributions. (Technically the result in [22] applies when the cancellations occur in the loop momentum polynomial rather than in the Feynman parameter polynomial, however by examining the zero mode integral, for example, in [17] one can see that a loop momentum representation exists for the string based rules and the result may be use.) In this section we will demonstrate how to apply unitarity to obtain the cuts in the one-loop amplitudes. For the four point $N = 8$

and $N = 6$ contributions the results agree completely with the string based results and for the remaining amplitudes the amplitudes are consistent with the cuts.

To calculate the cuts in all channels using the Cutkosky rules. Consider the regime where one of the momentum invariants is positive and the remainder are negative. This allow us to isolate the cuts in a single channel. We can then use the Cutkosky rules to obtain the cuts. To apply the cuts one needs explicit, preferable compact, expressions for tree amplitudes. In general, we consider the cut in the channel $(k_{m_1} + k_{m_1+1} + \dots + k_{m_2-1} + k_{m_2})^2$ for the loop amplitude $A_{n;1}(1, 2, \dots, n)$, depicted in fig. 1 and given by

$$\frac{i}{2} \int d\text{LIPS}(-\ell_1, \ell_2) A^{\text{tree}}(-\ell_1, m_1, \dots, m_2, \ell_2) A^{\text{tree}}(-\ell_2, m_2 + 1, \dots, m_1 - 1, \ell_1). \quad (5.1)$$

Instead of evaluating the phase-space integrals instead evaluate the off-shell integral

$$\frac{i}{2} \int \frac{d^D \ell_1}{(2\pi)^D} A^{\text{tree}}(-\ell_1, m_1, \dots, m_2, \ell_2) \frac{1}{\ell_2^2} A^{\text{tree}}(-\ell_2, m_2 + 1, \dots, m_1 - 1, \ell_1) \frac{1}{\ell_1^2} \Big|_{\text{cut}}. \quad (5.2)$$

whose cut is (5.1). This replacement is only valid in this channel. In evaluating this off-shell integral, we may substitute $\ell_1^2 = \ell_2^2 = 0$ in the numerator; any terms with ℓ_1^2 or ℓ_2^2 in the numerator cancels a cut propagator leading to an integral without a cut in this channel. Evaluating these cuts requires the tree amplitudes for all possible intermediate states, preferably in a compact form. For the four point those tree amplitudes which have been calculated previously in refs. [6,31] and in a helicity basis by Berends, Giele and Kuijf in ref [11] are sufficient. For pure gravity, the tree amplitudes $A^{\text{tree}}(- + + +)$ and $A^{\text{tree}}(+ + + +)$ vanish but $A^{\text{tree}}(- - + +)$ is non-zero and is given by

$$A^{\text{tree}}(1^-, 2^-, 3^+, 4^+) = \frac{\kappa^2}{4} \left(\frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} \right)^2 \times \frac{st}{u} \quad (5.3)$$

We first note that the $A^{1\text{-loop}}(-, +, +, +)$ and $A^{1\text{-loop}}(+, +, +, +)$ one loop amplitudes have no logarithms and hence no cuts. This is consistent with the fact that there are no non-vanishing pairs of tree amplitudes which could contribute to the Cutkosky rules.

To calculate the cuts in $A^{1\text{-loop}}(1^-, 2^-, 3^+, 4^+)$, first consider the cut in the s -channel, $(k_1 + k_2)^2$ as shown in fig. 2a.

$$\frac{i}{2} \int d\text{LIPS} A^{\text{tree}}(1^-, 2^-, \ell_2^+, \ell_1^+) \times A^{\text{tree}}(\ell_1^-, \ell_2^-, 3^+, 4^+) \quad (5.4)$$

This is non-zero for the case where the intermediate (cut) states are gravitons, however when the intermediate states are otherwise the tree amplitudes are zero. This is because the graviton vertex does not flip helicity of the fermions [11,32] hence the amplitude with two gravitons and two fermions of the same helicity vanishes, $A(g, g, \psi^+, \psi^+) = 0$. Similarly

the vector amplitudes and scalar amplitudes (with the concept of helicity being replaced by particle/antiparticle) vanish. Hence states other than gravitons do not contribute to eq. (5.4). For the supersymmetric decomposition this implies the s -channel cut will only be non-zero for the $N = 8$ contribution. Inserting the graviton tree amplitudes into (5.4) yields,

$$\frac{i\kappa^4}{32} \int d\text{LIPS} \left(\frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 \ell_2 \rangle \langle \ell_2 \ell_1 \rangle \langle \ell_1 1 \rangle} \right)^2 \times \frac{s(k_2 \cdot \ell_2)}{(k_1 \cdot \ell_2)} \left(\frac{\langle \ell_1 \ell_2 \rangle^4}{\langle 3 4 \rangle \langle 4 \ell_1 \rangle \langle \ell_1 \ell_2 \rangle \langle \ell_2 3 \rangle} \right)^2 \times \frac{s(k_3 \cdot \ell_2)}{(k_4 \cdot \ell_2)} \quad (5.5)$$

which we can rearrange, using the fact that ℓ_1 and ℓ_2 are onshell,

$$\frac{i\kappa^4}{32} s^2 \left(\frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 3 4 \rangle} \right)^2 \int d\text{LIPS} \frac{(k_2 \cdot \ell_2)}{(k_1 \cdot \ell_2)} \frac{(k_3 \cdot \ell_2)}{(k_4 \cdot \ell_2)} \frac{\langle \ell_1 \ell_2 \rangle^4}{\langle 2 \ell_2 \rangle^2 \langle \ell_1 1 \rangle^2 \langle 4 \ell_1 \rangle^2 \langle \ell_2 3 \rangle^2} \quad (5.6)$$

We can rearrange this, using

$$\begin{aligned} \frac{\langle \ell_1 \ell_2 \rangle^2}{\langle \ell_1 1 \rangle^2 \langle 2 \ell_2 \rangle^2} &= \frac{\langle \ell_1 \ell_2 \rangle^2 [1 2]^2}{\langle \ell_1 1 \rangle \langle 2 \ell_2 \rangle \langle \ell_1 1 \rangle [1 2] [1 2] \langle 2 \ell_2 \rangle} \\ &= \frac{\langle \ell_1 \ell_2 \rangle^2 [1 2]^2}{\langle \ell_1 1 \rangle \langle 2 \ell_2 \rangle \langle \ell_1 \ell_2 \rangle [\ell_2 2] [1 \ell_1] \langle \ell_1 \ell_2 \rangle} \\ &= \frac{[1 2]^2}{\langle 2 \ell_2 \rangle [\ell_2 2] [1 \ell_1] \langle \ell_1 1 \rangle} \\ &= \frac{[1 2]^2}{4(k_2 \cdot \ell_2)(k_1 \cdot \ell_1)} \end{aligned} \quad (5.7)$$

where we use $\langle \ell_1 1 \rangle [1 2] = -\langle \ell_1 \ell_2 \rangle [\ell_2 2]$ etc. Similarly

$$\frac{\langle \ell_1 \ell_2 \rangle^2}{\langle 4 \ell_1 \rangle^2 \langle \ell_2 3 \rangle^2} = \frac{[3 4]^2}{4(k_3 \cdot \ell_2)(k_4 \cdot \ell_1)} \quad (5.8)$$

Noting that

$$[1 2] [3 4] = \frac{-st}{\langle 2 3 \rangle \langle 4 1 \rangle} \quad (5.9)$$

we obtain the form of the cut

$$\begin{aligned} \frac{i\kappa^2}{8} A^{tree}(1^-, 2^-, 3^+, 4^+) s^3 t u \int d\text{LIPS} \frac{(k_2 \cdot \ell_2)}{(k_1 \cdot \ell_2)} \frac{(k_3 \cdot \ell_2)}{(k_4 \cdot \ell_2)} \times \frac{1}{16(k_2 \cdot \ell_2)(k_1 \cdot \ell_1)(k_3 \cdot \ell_2)(k_4 \cdot \ell_1)} \\ = \frac{i\kappa^2}{8} A^{tree}(1^-, 2^-, 3^+, 4^+) s^3 t u \int d\text{LIPS} \frac{1}{16(k_1 \cdot \ell_2)(k_4 \cdot \ell_2)(k_1 \cdot \ell_1)(k_4 \cdot \ell_1)} \end{aligned} \quad (5.10)$$

There is a useful identity, requiring the fact that both trees are on-shell.

$$\begin{aligned}
& \frac{1}{(k_1 \cdot \ell_2)(k_4 \cdot \ell_2)(k_1 \cdot \ell_1)(k_4 \cdot \ell_1)} \\
&= \frac{4}{s^2} \left(\frac{-1}{(k_1 \cdot \ell_1)(k_4 \cdot \ell_1)} + \frac{1}{(k_1 \cdot \ell_2)(k_4 \cdot \ell_1)} + \frac{1}{(k_1 \cdot \ell_1)(k_4 \cdot \ell_2)} + \frac{-1}{(k_1 \cdot \ell_2)(k_4 \cdot \ell_2)} \right) \\
&= \frac{16}{s^2} \left(\frac{1}{(k_1 - \ell_1)^2(k_4 + \ell_1)^2} + \frac{1}{(k_1 + \ell_2)^2(k_4 + \ell_1)^2} + \frac{1}{(k_1 - \ell_1)^2(k_4 - \ell_2)^2} + \frac{1}{(k_1 + \ell_2)^2(k_4 - \ell_2)^2} \right)
\end{aligned} \tag{5.11}$$

This uses momentum conservation extensively. Inserting the two propagators $1/\ell_1^2$ and $1/\ell_2^2$ and replacing $\int d\text{LIPS}$ by $\int d^D \ell / (2\pi)^D$ as in eq. (5.2) the cut in eq. (5.10) can now be recognised as the cut of the sum of two scalar box integrals with orderings 1234 and 2134 (the four terms above only correspond to two independent boxes.) These boxes have coefficients,

$$2 \frac{\kappa^2}{8} A^{\text{tree}}(1^-, 2^-, 3^+, 4^+)_{stu} \tag{5.12}$$

Next, consider cuts in the channel $(k_1 + k_4)^2$. These in general are more complex and also depend upon the multiplet under consideration. As can be seen from fig. 2b in this case all particles contribute. To evaluate the cut one needs the four point amplitudes with two external gravitons and two scalars or fermions or vectors or gravitinos. These may be obtained from the four graviton amplitudes using an extended form of the supersymmetric ward identities [30]. From these, we obtain

$$\begin{aligned}
A(g^-, \phi^-, \phi^+, g^+) &= \frac{\langle 13 \rangle^4}{\langle 12 \rangle^4} A(g^-, g^-, g^+, g^+) \\
A(g^-, \Lambda^-, \Lambda^+, g^+) &= \frac{\langle 13 \rangle^3}{\langle 12 \rangle^3} A(g^-, g^-, g^+, g^+) \\
A(g^-, A^-, A^+, g^+) &= \frac{\langle 13 \rangle^2}{\langle 12 \rangle^2} A(g^-, g^-, g^+, g^+) \\
A(g^-, \psi^-, \psi^+, g^+) &= \frac{\langle 13 \rangle}{\langle 12 \rangle} A(g^-, g^-, g^+, g^+)
\end{aligned} \tag{5.13}$$

where $(g, \psi, A, \Lambda, \phi)$ are the members of the $N = 8$ multiplet.

It is useful to count each states contribution to the cut relative to that for a scalar. All these states will contribute to the cut,

$$\frac{i}{2} \int d\text{LIPS} A(4^+, 1^-, \phi(\ell_2)^-, \phi(\ell_1)^+) \times A(\phi(\ell_1)^-, \phi(\ell_2)^+, 2^-, 3^+) \times \rho_{N=8} \tag{5.14}$$

which is explicitly

$$\frac{i\kappa^4}{32} \frac{t^2}{\langle 23 \rangle^2 \langle 41 \rangle^2} \int d\text{LIPS} \frac{\langle 1 \ell_1 \rangle^4 \langle 2 \ell_1 \rangle^4 \langle 1 \ell_2 \rangle^2 \langle 2 \ell_2 \rangle^2}{\langle 3 \ell_1 \rangle^2 \langle 4 \ell_1 \rangle^2 \langle \ell_1 \ell_2 \rangle^4} \frac{(k_1 \cdot \ell_2)(k_2 \cdot \ell_2)}{(k_1 \cdot \ell_1)(k_2 \cdot \ell_1)} \rho_{N=8} \tag{5.15}$$

The factor ρ for $N = 8$ will be

$$\rho_{N=8} = x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - 56x^{-2} + 28x^{-4} - 8x^{-6} + x^{-8} \quad (5.16)$$

where

$$x^2 = \frac{\langle 1 \ell_2 \rangle \langle 2 \ell_1 \rangle}{\langle 1 \ell_1 \rangle \langle 2 \ell_2 \rangle} \quad (5.17)$$

The central term, 70, arises from the 35 complex scalars, the x^2 and x^{-2} from the Weyl fermions, the x^4 and x^{-4} from the vectors, the x^6 and x^{-6} from the gravitinos and the x^8 and x^{-8} from the gravitons. These relative weights compared to the scalar contribution are obtained from eq. (5.13). This simplifies to

$$\begin{aligned} \rho_{N=8} &= (x - x^{-1})^8 = \frac{(x^2 - 1)^8}{x^8} = \frac{(\langle 1 \ell_2 \rangle \langle 2 \ell_1 \rangle - \langle 1 \ell_1 \rangle \langle 2 \ell_2 \rangle)^8}{\langle 1 \ell_2 \rangle \langle 2 \ell_1 \rangle \langle 1 \ell_1 \rangle \langle 2 \ell_2 \rangle^4} \\ &= \frac{\langle 1 2 \rangle^8 \langle \ell_1 \ell_2 \rangle^8}{(\langle 1 \ell_2 \rangle \langle 2 \ell_1 \rangle \langle 1 \ell_1 \rangle \langle 2 \ell_2 \rangle)^4} \end{aligned} \quad (5.18)$$

using the identity $\langle a b \rangle \langle c d \rangle = \langle a c \rangle \langle b d \rangle + \langle a d \rangle \langle c b \rangle$. The cut then becomes

$$\frac{i\kappa^4}{32} t^2 \frac{\langle 1 2 \rangle^8}{\langle 2 3 \rangle^2 \langle 4 1 \rangle^2} \int d\text{LIPS} \frac{(k_1 \cdot \ell_2)(k_2 \cdot \ell_2)}{(k_1 \cdot \ell_1)(k_2 \cdot \ell_1)} \frac{\langle \ell_1 \ell_2 \rangle^4}{(\langle 1 \ell_2 \rangle \langle \ell_1 4 \rangle \langle \ell_2 2 \rangle \langle 3 \ell_1 \rangle)^2} \quad (5.19)$$

Similarly to before we have the identities

$$\frac{\langle \ell_1 \ell_2 \rangle^2}{\langle \ell_1 4 \rangle^2 \langle 1 \ell_2 \rangle^2} = \frac{[4 1]^2}{4(k_1 \cdot \ell_2)(k_4 \cdot \ell_1)}; \quad \frac{\langle \ell_1 \ell_2 \rangle^2}{\langle 3 \ell_1 \rangle^2 \langle \ell_2 2 \rangle^2} = \frac{[2 3]^2}{4(k_2 \cdot \ell_2)(k_3 \cdot \ell_1)} \quad (5.20)$$

and

$$[4 1]^2 [2 3]^2 = \frac{t^2 s^2}{\langle 1 2 \rangle^2 \langle 3 4 \rangle^2} \quad (5.21)$$

which give the cut to be

$$\frac{i\kappa^4}{32} t^4 s^2 \left(\frac{\langle 1 2 \rangle^8}{\langle 1 2 \rangle^2 \langle 2 3 \rangle^2 \langle 3 4 \rangle^2 \langle 4 1 \rangle^2} \right) \int d\text{LIPS} \frac{(k_1 \cdot \ell_2)(k_2 \cdot \ell_2)}{(k_1 \cdot \ell_1)(k_2 \cdot \ell_1)} \times \frac{1}{16(k_1 \cdot \ell_2)(k_4 \cdot \ell_1)(k_2 \cdot \ell_2)(k_3 \cdot \ell_1)} \quad (5.22)$$

which is

$$\frac{i\kappa^2}{8} A^{\text{tree}}(4^+, 1^-, 2^-, 3^+) t^2 s u \int d\text{LIPS} \frac{1}{16(k_1 \cdot \ell_1)(k_2 \cdot \ell_1)(k_4 \cdot \ell_1)(k_3 \cdot \ell_1)} \quad (5.23)$$

This is just the analogue of the previous case. We can carry out the same factorization as before and find the two appropriate boxes with ordering 1234 and 1243. The cuts in the $(k_1 + k_3)^2$ channel are obtained exactly as the t -channel cuts. Replacing, $\int d\text{LIPS}$ by

$\int d^D \ell / (2\pi)^2$ we can thus deduce that the cuts in all channels are described by the sum over the scalar boxes with coefficients $i\kappa^2 A^{\text{tree}} stu/4$. The scalar box, for ordering of legs 1234 is

$$\begin{aligned} I_4 &= \frac{r_\Gamma(4\pi)^\epsilon}{(4\pi)^2} \frac{1}{st} \left\{ \frac{2}{\epsilon} \left[(-s)^{-\epsilon} + (-t)^{-\epsilon} \right] - \ln^2(-s/-t) - \pi^2 \right\} \\ &= \frac{r_\Gamma(4\pi)^\epsilon}{(4\pi)^2} \frac{1}{st} \left\{ \frac{4}{\epsilon^2} - \frac{2}{\epsilon} \left(\ln(-s) + \ln(-t) \right) + 2 \ln(-s) \ln(-t) - \pi^2 \right\} \end{aligned} \quad (5.24)$$

This sum over boxes then evaluates to

$$\begin{aligned} A^{N=8}(1^-, 2^-, 3^+, 4^+) &= i\kappa^2 \frac{r_\Gamma(4\pi)^\epsilon}{(4\pi)^2} A^{\text{tree}}(1^-, 2^-, 3^+, 4^+) \\ &\times \left(\frac{2s \ln(-s) + 2t \ln(-t) + 2u \ln(-u)}{\epsilon} + 2s \ln(-t) \ln(-u) + 2t \ln(-u) \ln(-s) + 2u \ln(-s) \ln(-t) \right) \end{aligned} \quad (5.25)$$

Since we have produced an expression with the correct cuts which is written in terms of integral functions we can use the results of ref. [21,22] to deduce that this expression is the entire amplitude. This is in agreement with the explicit calculation of the previous section.

For the other cases, the, $N = 6, N = 4, N = 2$ and $N = 0$ contributions, we may perform a similar calculation to the $N = 8$ case but with differing ρ . We will have

$$\begin{aligned} \rho_{N=6} &= (x - x^{-1})^6 = \frac{\langle 1 2 \rangle^6 \langle \ell_1 \ell_2 \rangle^6}{(\langle 1 \ell_2 \rangle \langle 2 \ell_1 \rangle \langle 1 \ell_1 \rangle \langle 2 \ell_2 \rangle)^3} \\ \rho_{N=4} &= (x - x^{-1})^4 = \frac{\langle 1 2 \rangle^4 \langle \ell_1 \ell_2 \rangle^4}{(\langle 1 \ell_2 \rangle \langle 2 \ell_1 \rangle \langle 1 \ell_1 \rangle \langle 2 \ell_2 \rangle)^2} \\ \rho_{N=2} &= (x - x^{-1})^2 = \frac{\langle 1 2 \rangle^2 \langle \ell_1 \ell_2 \rangle^2}{(\langle 1 \ell_2 \rangle \langle 2 \ell_1 \rangle \langle 1 \ell_1 \rangle \langle 2 \ell_2 \rangle)} \end{aligned} \quad (5.26)$$

A similar calculation of the cuts may be performed. For the $N = 6$ calculation the cuts are enough to completely reconstruct the amplitude and we obtain an amplitude in agreement with the explicit string based-calculation. For the other case, the cuts would not be enough to completely specify the amplitude but provide strong consistency checks on the amplitudes. In general, the cuts are relatively simple to calculate when compact forms for the tree amplitudes exist and where the cuts specify the amplitude completely they would be the calculational method of choice.

In the next section we will illustrate the use of the cuts to obtain the logarithmic parts of two-loop amplitude $A(+, +, +, +)$.

6. The Cuts in $A^{2\text{-loop}}(+, +, +, +)$

In this section we will use the Cutkosky rules to calculate the cuts in the two-loop pure gravity amplitude $A^{2\text{-loop}}(+, +, +, +)$. It is possible to do this because of the simple form

of the one-loop amplitude $A^{1\text{-loop}}(+, +, +, +)$. If we consider the two-loop cuts we must consider the cuts in the three-particle intermediate states as shown in fig. 3a. Fortunately, these cuts vanish because one of the two tree amplitudes must inevitably have a single negative helicity and this tree vanishes. Considering the remaining possibilities we find, for example in the s -channel that the configurations in figs. 3b and 3c may contribute. In this case we have a product of $A^{\text{tree}}(-, -, +, +)$ and $A^{1\text{-loop}}(+, +, +, +)$ neither of which vanish. Since the one-loop amplitude does not contain logarithms or dilogarithms the evaluation of the cut is analogous to calculating a one-loop cut and we are able to do so. Explicitly the s -channel cut in fig. 3b is

$$\frac{i}{2} \int d\text{LIPS}(-\ell_1, \ell_2) A^{1\text{-loop}}(1^+, 2^+, -\ell_1^+, \ell_2^+) A^{\text{tree}}(-\ell_2^-, \ell_1^-, 3^+, 4^+), \quad (6.1)$$

where $d\text{LIPS}(-\ell_1, \ell_2)$ denotes the Lorentz-invariant phase space measure. The explicit form of the tree and one-loop amplitudes is obtainable from eq. (4.2) and eq. (5.3). With the parameterisation shown

$$\begin{aligned} A^{1\text{-loop}}(1^+, 2^+, -\ell_1^+, \ell_2^+) &= \frac{-2i\kappa^4}{(4\pi)^2} \frac{1}{1920} \left(\frac{s(2k_2 \cdot \ell_2)}{\langle 12 \rangle \langle 2\ell_2 \rangle \langle \ell_2 \ell_1 \rangle \langle \ell_1 1 \rangle} \right)^2 \times \left(s^2 + s(2k_2 \cdot \ell_2) + (2k_2 \cdot \ell_2)^2 \right) \\ A^{\text{tree}}(-\ell_2^-, \ell_1^-, 3^+, 4^+) &= \frac{i\kappa^2}{4} \left(\frac{\langle \ell_2 \ell_1 \rangle^3}{\langle \ell_2 3 \rangle \langle 3 4 \rangle \langle 4 \ell_1 \rangle} \right)^2 \frac{s(2k_3 \cdot \ell_2)}{(2k_4 \cdot \ell_2)} \end{aligned} \quad (6.2)$$

The evaluation of this cut employs many of the tricks already used to obtain checks of the cuts for the one-loop results. Using eq. (5.7) and (5.8) we can simplify the cut to

$$\sim \frac{1}{1920} \frac{s^5 t^2}{(\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle)^2} \int d\text{LIPS} \frac{(2k_2 \cdot \ell_2) \left(s^2 + s(2k_2 \cdot \ell_2) + (2k_2 \cdot \ell_2)^2 \right)}{(2k_1 \cdot \ell_1)(2k_4 \cdot \ell_1)(2k_4 \cdot \ell_2)} \quad (6.3)$$

Using

$$\frac{1}{(2k_4 \cdot \ell_1)(2k_4 \cdot \ell_2)} = \frac{1}{s} \left(\frac{1}{(2k_4 \cdot \ell_1)} - \frac{1}{(2k_4 \cdot \ell_2)} \right) = \frac{1}{s} \left(\frac{1}{(k_4 + \ell_1)^2} + \frac{1}{(k_3 + \ell_1)^2} \right) \quad (6.4)$$

this reduces to a sum of two box integrals. However since $(2k_2 \cdot \ell_2) = -(2k_1 \cdot \ell_1)$ the box integrands further reduce to the triangle integrals shown in fig. 4 with loop momentum polynomial

$$\left(s^2 + s(2k_2 \cdot \ell_2) + (2k_2 \cdot \ell_2)^2 \right) \quad (6.5)$$

Evaluating this integral gives the result,

$$\left(\frac{s^2 + t^2 + u^2}{2\epsilon^2} + \frac{1}{2\epsilon} (3u^2 + 3t^2 - 2s^2) + \frac{1}{4} (7u^2 + 7t^2 - 5s^2) \right) \times (-s)^{-1-\epsilon} \quad (6.6)$$

The second triangle gives the same result and the contribution from fig. 3c is also the same. This equation has the correct cut in the s -channel. The t and u channel cuts are obtained from this by permuting s, t and u . We can thus obtain an expression for the logarithmic parts of $A^{2\text{-loop}}(1^+, 2^+, 3^+, 4^+)$

$$\begin{aligned}
A^{2\text{-loop}}(1^+, 2^+, 3^+, 4^+) \sim & A^{1\text{-loop}}(1^+, 2^+, 3^+, 4^+) \times \\
& \left(\frac{(-s)^{1-\epsilon} + (-t)^{1-\epsilon} + (-u)^{1-\epsilon}}{2\epsilon^2} + \frac{1}{2} \frac{(3u^2 + 3t^2 - 2s^2)}{(s^2 + t^2 + u^2)} \frac{(-s)^{1-\epsilon}}{\epsilon} \right. \\
& + \frac{1}{2} \frac{(3u^2 + 3s^2 - 2t^2)}{(s^2 + t^2 + u^2)} \frac{(-t)^{1-\epsilon}}{\epsilon} + \frac{1}{2} \frac{(3s^2 + 3t^2 - 2u^2)}{(s^2 + t^2 + u^2)} \frac{(-u)^{1-\epsilon}}{\epsilon} \\
& \left. + \text{polynomials} \right)
\end{aligned} \tag{6.7}$$

where the polynomial pieces are not obtainable from the cuts. Associated with the logarithms in the above expression are $1/\epsilon$ poles. The $A(++++)$ two-loop amplitude is an interesting object in perturbative gravity. Gravity is non-renormalisable at two loops however not all amplitudes contain obvious non-renormalisable UV infinities. As is well known [33], the two loop amplitude $A(--++)$ does not contain such infinities and the infinities reside in the so-called ‘‘helicity-flip’’ amplitudes $A(++++)$ and $A(-+++)$. Calculation of the UV infinities for these amplitudes is a considerable undertaking and the explicit verification of the non-renormalisability of gravity at two-loops was an important but difficult calculation [34,35]. Although we have a suggested form for the infinities in this amplitude, obtained with an almost trivial calculation, we are not able to extract the UV infinity. Specifically the recognising of the UV and IR infinities is not possible by examining the cut integrals. When the cut was reduced to a triangle integral with polynomial in eq. (6.5) one would normally extract the UV infinity from the $\ell_\mu \ell_\nu$ which would give

$$\frac{1}{\epsilon} \times \delta_{\mu\nu} \tag{6.8}$$

However this yields zero since the coefficient is $k_1^\mu k_1^\nu$. Since the cut is only sensitive up to terms proportional to ℓ_1^2 and ℓ_2^2 the loop momentum polynomial could have been replaced by

$$\left((2k_1 \cdot \ell_2)^2 + (2k_1 \cdot \ell_2)(2k_2 \cdot \ell_2) + (2k_2 \cdot \ell_2)^2 \right) \tag{6.9}$$

without affecting the cut. However this polynomial would give an UV infinity since the δ -function would no longer vanish for the middle term. Thus although tempting, we are unable to deduce the coefficients of the UV infinities in the two-loop amplitude, at least without further information.

For the configuration $A(1^-, 2^+, 3^+, 4^+)$ the form of the one-loop amplitude also does not contain logarithms and one may evaluate the equivalent cut diagrams to those in fig. 3b

and 3c however in this case there is a diagram as depicted in fig. 5 which is non-zero and needs genuine two-loop integrals to evaluate.

Two-loop amplitudes are formidable calculations in general. Progress towards a string based or string inspired method has been made both based upon the infinite string tension limit [36] and upon the world-line formalism [37]. In any formalism we expect unitarity to provide very useful checks upon the calculations as evidenced by the simplicity by which we obtained the cuts in $A^{2\text{-loop}}(++++)$ (albeit the simplest case possible).

7. Conclusions

Recently new rules for calculations of one-loop amplitudes have been constructed using string theory methods. These have been successfully used to do calculations in both gauge theory and gravity which have not been practical using conventional methods. In this paper we gave a detailed description of a set of rules for gravity derived from string theory. We used the rules to calculate one-loop amplitudes for four graviton scattering. These covered theories with arbitrary particles content. Our results are consistent both with previous calculations and formal arguments concerning divergences. Although perturbative quantum gravity is a non-renormalisable field theory the UV divergences do not appear in our calculations.

In order to simplify the calculations we used a supersymmetric decomposition of the amplitudes also inspired by string theory. In a supersymmetric theory, provided one uses a suitable formalism, there are generally a large number of cancellations between different particle amplitudes. Careful choice of the supersymmetric multiplet amplitudes calculated enabled us to exploit these simplifications. Individual particle contributions could then be found from linear combinations of these supersymmetric amplitudes. The supersymmetric decomposition in the string based rules has proved extremely useful since it reduces the degree of the momentum loop polynomial diagram by diagram. Such cancellations do not occur even in a normal superfield formalism but are familiar if one uses a background field superfield formalism. The calculational advantages of calculating S -matrix elements using background field methods are widespread especially in situations where many pure gauge vertices appear or in situations where cancellations are possible.

We used unitarity constraints to check the amplitudes calculated. These constraints were found by use of the Cutkosky rules. For the $N = 8$ and $N = 6$ multiplets unitarity determined the amplitudes completely. For the remaining cases those parts of the amplitudes containing cuts could be checked. Since the loop momentum polynomial grows as $2n$ for an n -point amplitude rather than as n for gauge theories the Cutkosky rules are not as powerful a constraint in perturbative gravity as in gauge theories.

We also showed how the Cutkosky method could be used to obtain the logarithmic parts of a two loop gravity amplitude by calculating the cuts for the two-loop $(+, +, +, +)$

helicity amplitude. This amusing calculation is incomplete since there are potential polynomial terms but it does illustrate the potency of unitarity.

In conclusion we have found considerable calculational benefits from calculating using the string-based methods [2,3] and derivatives thereof [21,22]. Many of the techniques motivated by string theory have proved useful in these gravity calculations and we expect these to have wider validity.

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Figure Captions

1 *A generic cut in the amplitude is given by an integral over lorentz-invariant phase space (LIPS) of the product of two tree amplitudes.*

2 *The helicity configurations for the s and t cuts in $A^{1-loop}(+, +, +, +)$*

3 *The three possible contributions to the s -channel cut for $A^{2-loop}(+, +, +, +)$. Contribution (a) is vanishing since there is no choice of helicity for the intermediate legs where both trees are non-vanishing.*

4 *After manipulations, the s -channel cut in $A^{2-loop}(+, +, +, +)$ reduces to a sum of simple triangle integrals with integrands quadratic in the loop momentum.*

5 *This helicity configuration is non-vanishing and contributes to the cuts in $A^{2-loop}(-, +, +, +)$.*

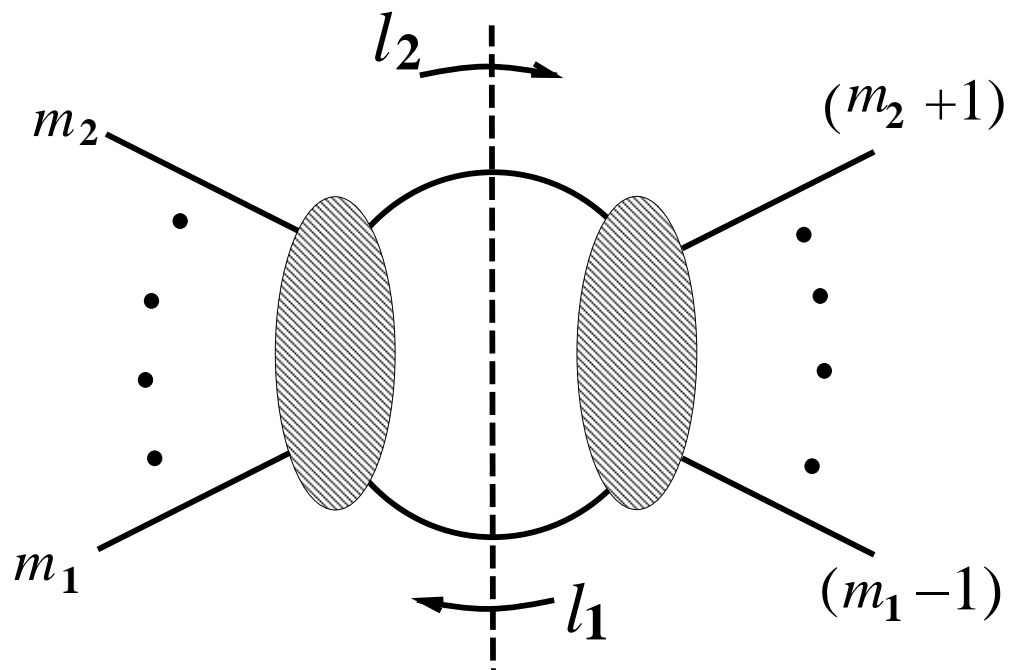
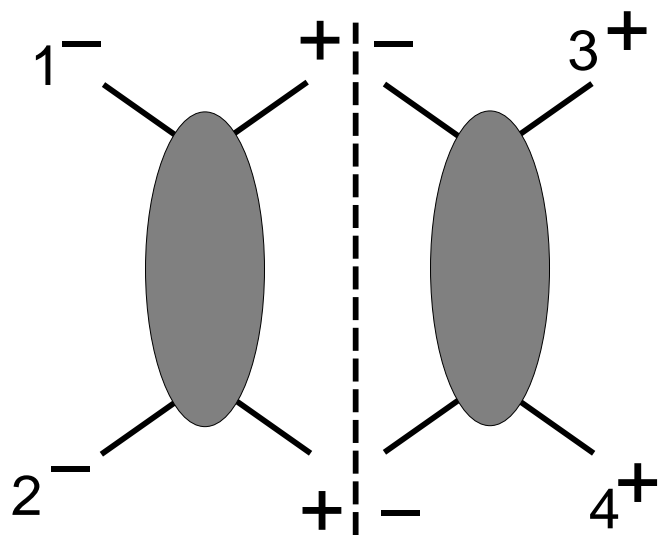


Fig. 1

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(a)



(b)

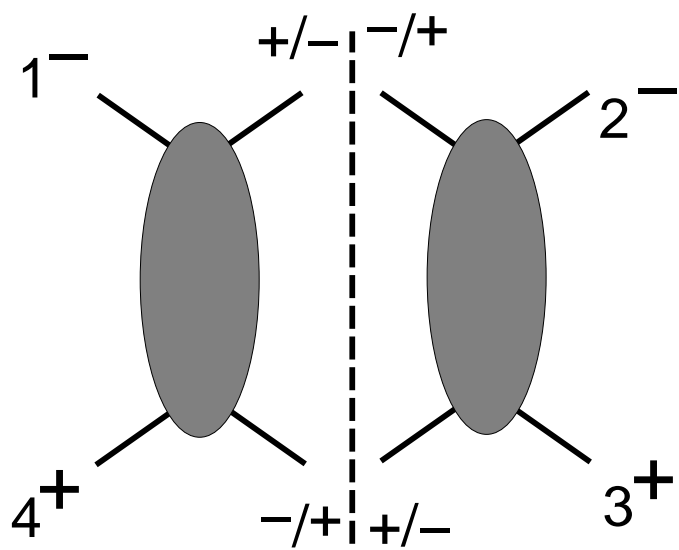


Fig. 2

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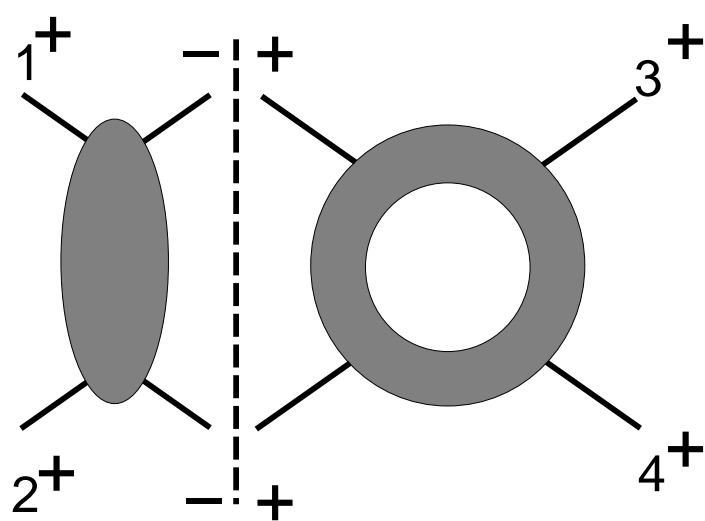
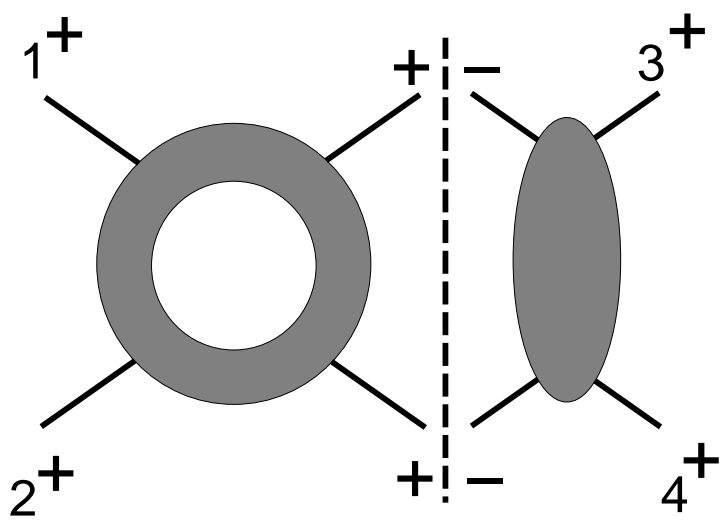
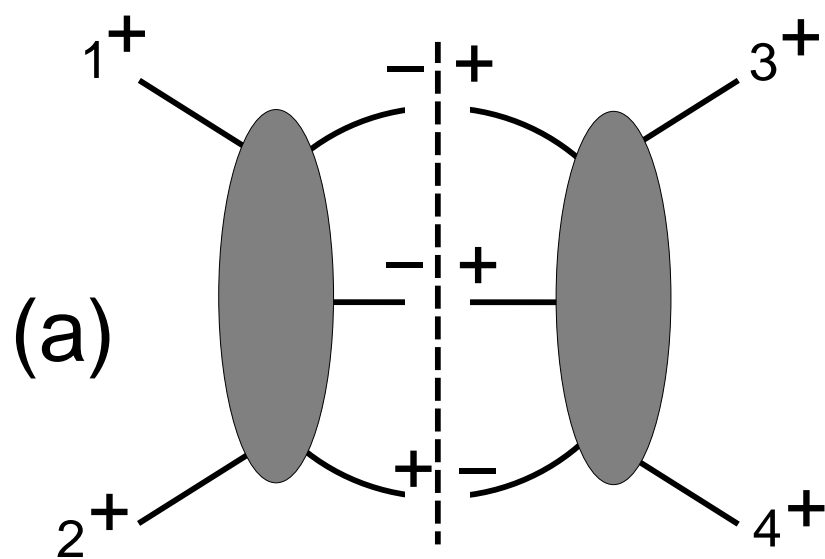


Fig. 3

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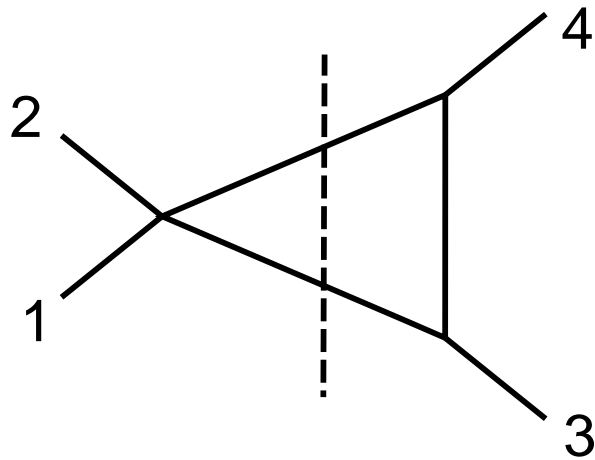
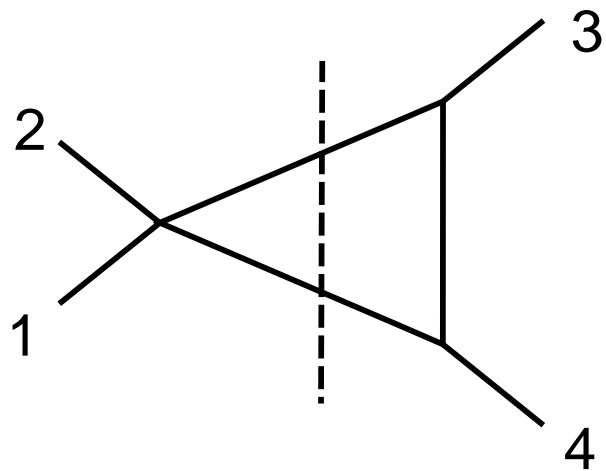


Fig. 4

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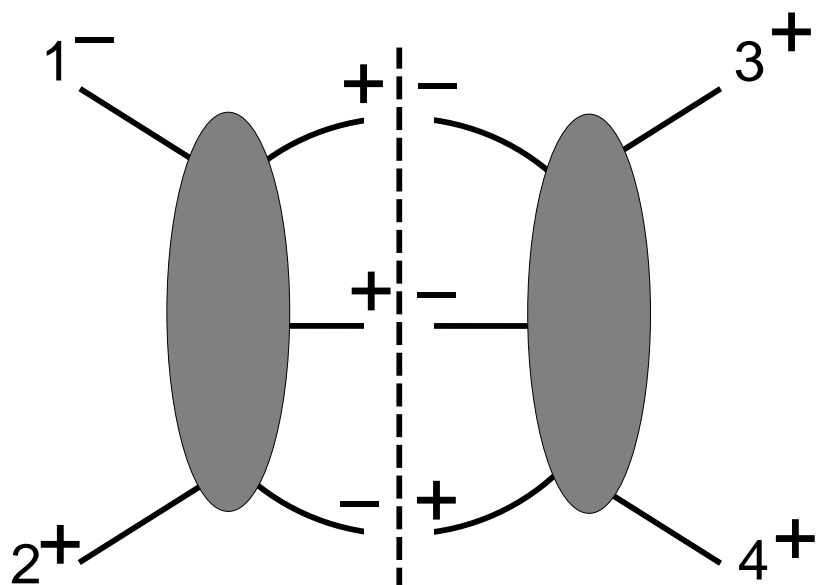


Fig. 5

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